Characterizations of symmetric cones by means of the basic relative invariants

Hideto Nakashima

Kyushu University

2015/6/24

RIMS, Kyoto university

Background

$\boldsymbol{V = \mathrm{Sym}(r, \mathbb{R})}$,
$\boldsymbol{\Omega = \mathrm{Sym}(r, \mathbb{R})^{++}}$,
$\boldsymbol{W = V_{\mathbb{C}}\ (= \mathrm{Sym}(r, \mathbb{C}))}$
$\boldsymbol{\Delta_1(w), \ldots, \Delta_r(w)}$: the principal minors of $\boldsymbol{w \in W}$
$\boldsymbol{T_\Omega := \Omega + iV}$: Tube domain

Classical fact

Put $\boldsymbol{\Delta_0(w) = 1}$. If $\boldsymbol{w \in T_\Omega}$, then one has

$$\boldsymbol{\mathrm{Re}\,\frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r).}$$

→This result can be generalize to any irreducible symmetric cone.

(Ishi–Nomura 2008)

Background

V : simple Euclidean Jordan algebra
Ω : irreducible symmetric cone of V
$T_\Omega := \Omega + iV \subset W = V_{\mathbb{C}}$
$\Delta_1(x), \ldots, \Delta_r(x)$: the principal minors of V
$\rightarrow$ naturally continued to holomorphic polynomial functions of W

Theorem A (Ishi–Nomura 2008)

Put $\Delta_0(w) = 1$. If $w \in T_\Omega$, then one has

$$\mathrm{Re}\,\frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r).$$

Q. Does this property characterize symmetric cones?
A. No (Ishi-Nomura 2008)
Q. How does this property generalize to homogeneous cones?
$\rightarrow$ Today's topic

Talking plan

(1) Background
  (i) Theorem A
(2) Generalization of Theorem A
  (i) Setting and definitions
  (ii) matrix realization of homogeneous cones
  (iii) known results
  (iv) Theorem 1 (generalization of Theorem A)
(3) Characterization of symmetric cones
  (i) dual cones
  (ii) Main theorem (characterization of symmetric cones)
  (iii) sketch of the proof

Setting

V: finite-dimensional real vector space
Ω: open convex cone in V containing no entire line
$G(\Omega) := \{g \in GL(V);\ g(\Omega) = \Omega\}$

$$\Omega \text{ is homogeneous} \Leftrightarrow G(\Omega) \text{ acts on } \Omega \text{ transitively}$$

Assume that Ω is homogeneous

$\exists H$: split solvable Lie subgroup of $G(\Omega)$ s.t.

$$H \curvearrowright \Omega: \text{simply transitively.}$$

Example

$\mathcal{S}_N = \mathrm{Sym}(N, \mathbb{R})$
$\mathcal{S}_N^+ = \mathrm{Sym}(N, \mathbb{R})^{++} = \{x \in V;\ x \text{ is positive definite}\}$
$g \in GL(N, \mathbb{R})$ acts on $\mathcal{S}_N^+$ by $g \cdot x := gx\,{}^tg$.
$\mathcal{H}_N$: group of lower triangular matrices with positive diagonals.
$\to \mathcal{H}_N$ acts on $\mathcal{S}_N^+$ simply transitively

Matrix realization of homogeneous cones (Ishi 2006)

$N = n_1 + \cdots + n_r$: partition of $N \in \mathbb{N}$

$\mathcal{V}_{lk} \subset \mathrm{Mat}(n_l, n_k; \mathbb{R})$: system of vector spaces satisfying

(V0) $\mathcal{V}_{jj} = \mathbb{R} I_{n_j}$ $(j = 1, \ldots, r)$,

(V1) $A \in \mathcal{V}_{lk}$, $B \in \mathcal{V}_{kj} \Rightarrow AB \in \mathcal{V}_{lj}$ $\quad (j < k < l)$,

(V2) $A \in \mathcal{V}_{lj}$, $B \in \mathcal{V}_{kj} \Rightarrow A\,{}^tB \in \mathcal{V}_{lk}$ $\quad (j < k < l)$,

(V3) $A \in \mathcal{V}_{kj} \Rightarrow A\,{}^tA \in \mathcal{V}_{kk}$ $\quad (j < k)$.

$$\mathcal{Z}_{\mathcal{V}} = \left\{ X = \begin{pmatrix} X_{11} & {}^tX_{21} & \cdots & {}^tX_{r1} \\ X_{21} & X_{22} & \ddots & {}^tX_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{pmatrix} ; \begin{array}{l} X_{kk} = x_{kk} I_{n_k}, \\ \quad (x_{kk} \in \mathbb{R}) \\ X_{lk} \in \mathcal{V}_{lk} \end{array} \right\} \subset \mathcal{S}_N,$$

$\mathcal{P}_{\mathcal{V}} = \{ X \in \mathcal{Z}_{\mathcal{V}};\ X \text{ is positive definite} \}$.

$\to \mathcal{P}_{\mathcal{V}}$ is a homogeneous cone of rank r.

Any homogeneous cone Ω can be realized as some $\mathcal{P}_{\mathcal{V}}$.

Split solvable Lie subgroup H

H is linearly isomorphic to

$$\left\{ h = \begin{pmatrix} T_{11} & & & \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \cdots & T_{rr} \end{pmatrix}; \quad \begin{matrix} T_{kk} = e^{t_k/2} I_{n_k} \\ (t_k \in \mathbb{R}) \\ T_{lk} \in \mathcal{V}_{lk} \end{matrix} \right\} \subset \mathcal{H}_N.$$

The action on $\mathcal{P}_{\mathcal{V}}$ is described as $h \cdot x = hx\,{}^t h$.

Define. f: relatively H-invariant function of Ω

$$\exists \chi \colon H \to \mathbb{R}\text{: 1-dim. rep. s.t. } f(h \cdot x) = \chi(h) f(x).$$

$\to \exists \underline{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ s.t. $\chi(h) = e^{\nu_1 t_1 + \cdots + \nu_r t_r}$ (multiplier).
In particular we have

$$f(\mathrm{diag}(x_1, \ldots, x_r)) = x_1^{\nu_1} \cdots x_r^{\nu_r}$$

Basic relative invariants

Theorem (Ishi–Nomura 2008)

There exist just r relatively H-invariant irreducible polynomials $\Delta_1(x), \ldots, \Delta_r(x)$, and Ω is described as

$$\Omega = \{x \in V;\ \Delta_1(x) > 0, \ldots, \Delta_r(x) > 0\}.$$

$\to$ Δ_j's are called the basic relative invariants of Ω.
$\underline{\sigma}_j = (\sigma_{j1}, \ldots, \sigma_{jr})$: multiplier of $\Delta_j(x)$

$$\sigma := \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk})_{1 \le j,k \le r}: \text{ multiplier matrix}$$

multiplier matrix is lower, the diagonal elements are all 1 (Ishi 2001).
We have an algorithm for calculating σ (N-. 2014).

Complexification

$W := V_{\mathbb{C}}$, and $T_{\Omega} := \Omega + iV$.
$H_{\mathbb{C}}$: complexification of H.
$f_{\mathbb{C}}$: complexification of a relatively H-invariant function f.
relatively $H_{\mathbb{C}}$-invariance

$$\text{“}f_{\mathbb{C}}(\rho(h)w) = \chi(h) f_{\mathbb{C}}(w) \text{ for } \forall h \in H_{\mathbb{C}},\ w \in W\text{”}$$

$\rightarrow \exists h'$ s.t. $\rho(h)w = \rho(h')w$ for $\forall w \in W$, but $\chi(h) \stackrel{?}{=} \chi(h')$.

If f is rational, then χ is well-defined.

$\Delta_1, \ldots, \Delta_r$: naturally continued to holomorphic poly.

$$\mathcal{S} := \{w \in W;\ \exists \Delta_j(w) = 0\}.$$

Put $N_{\mathbb{C}} := \{h \in H_{\mathbb{C}};\ \mathbf{diag} = I_{n_j}\ (j = 1, \ldots, r)\}$.

Then
$$f_{\mathbb{C}}(n \cdot w) = f_{\mathbb{C}}(w),$$
$$f_{\mathbb{C}}(\mathbf{diag}(x_1, \ldots, x_r)) = x_1^{\nu_1} \cdots x_r^{\nu_r}.$$

Known results

Proposition (Ishi-Nomura 2008)

(i) For any $w \in W \backslash \mathcal{S}$, there exist unique $n \in N_{\mathbb{C}}$ and $\alpha_j(w) \in \mathbb{C}^\times$ $(j = 1, \ldots, r)$ such that

$$w = n \cdot \mathrm{diag}(\alpha_1(w), \ldots, \alpha_r(w)).$$

(ii) If $w \in T_\Omega$, then one has $\mathrm{Re}\, \alpha_k(w) > 0$ for $k = 1, \ldots, r$.

$\rightarrow$ Describe $\alpha_1(w), \ldots, \alpha_r(w)$ by using $\Delta_1(w), \ldots, \Delta_r(w)$.

For $\underline{\mu}$, $\underline{\tau} \in \mathbb{Z}^r$, put

$$\alpha^{\underline{\mu}}(w) := \alpha_1(w)^{\mu_1} \cdots \alpha_r(w)^{\mu_r},$$

$$\Delta^{\underline{\tau}}(w) := \Delta_1(w)^{\tau_1} \cdots \Delta_r(w)^{\tau_r},$$

$$\underline{e}_j := (0, \ldots, 0, \overset{j}{\check{1}}, 0, \ldots, 0).$$

Generalization of Theorem A

Theorem 1

Let $w \in T_\Omega$. Then one has $\alpha_j(w) = \Delta^{\underline{e}_j \sigma^{-1}}(w)$, and hence

$$\mathrm{Re}\,\Delta^{\underline{e}_j \sigma^{-1}}(w) > 0 \quad (j = 1, \ldots, r).$$

proof. For each j, we have

$$\begin{aligned}
\Delta_j(w) &= \Delta_j\left(n \cdot \mathrm{diag}(\alpha_1(w), \ldots, \alpha_r(w))\right) \\
&= \Delta_j\left(\mathrm{diag}(\alpha_1(w), \ldots, \alpha_r(w))\right) \\
&= \alpha_1(w)^{\sigma_{j1}} \cdots \alpha_r(w)^{\sigma_{jr}} \\
&= \alpha^{\underline{\sigma}_j}(w).
\end{aligned}$$

$$\to \Delta^{\underline{\tau}}(w) = (\alpha^{\underline{\sigma}_1}(w))^{\tau_1} \cdots (\alpha^{\underline{\sigma}_r}(w))^{\tau_r} = \alpha^{\underline{\tau}\sigma}(w).$$

Thus we have $\alpha_j(w) = \alpha^{\underline{e}_j}(w) = \Delta^{\underline{e}_j \sigma^{-1}}(w)$.

Case of symmetric cones

V: Euclidean Jordan algebra
$\Delta_j(x)$: principal minors of V

In this case we have $\underline{\sigma}_j = (1, \dots, \overset{j}{1}, 0, \dots, 0)$ and hence

$$\sigma = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & 1 & 1 \end{pmatrix} \to \sigma^{-1} = \begin{pmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix}$$

Thus Theorem 1 leads us to the known result:
If $w \in \Omega + iV$, then one has

$$\begin{aligned} \mathrm{Re}\, \Delta^{\underline{e}_j \sigma^{-1}}(w) &= \mathrm{Re}\, \Delta_{j-1}(w)^{-1} \Delta_j(w) \\ &= \mathrm{Re}\, \frac{\Delta_j(w)}{\Delta_{j-1}(w)} > 0. \end{aligned}$$

Talking plan

(1) Background
   (i) Theorem A
(2) Generalization of Theorem A
   (i) Setting and definitions
   (ii) matrix realization of homogeneous cones
   (iii) known results
   (iv) Theorem 1 (generalization of Theorem A)
(3) Characterization of symmetric cones
   (i) dual cones
   (ii) Main theorem (characterization of symmetric cones)
   (iii) sketch of the proof

Dual cone

Ω: homogeneous cone in V
$\langle\cdot|\cdot\rangle$: inner product of V
Dual cone Ω^* of Ω is defined to be

$$\Omega^* := \left\{x \in V;\ \langle x\,|\,y\rangle > 0 \text{ for all } y \in \overline{\Omega}\backslash\{0\}\right\}.$$

$\Delta_1^*(x), \ldots, \Delta_r^*(x)$: basic relative invariants of Ω^*
the index is determined as the multiplier matrix σ_*
to be upper triangular

Ω is irreducible $\Leftrightarrow$ $\Omega = \Omega_1 \oplus \Omega_2$ implies $\Omega_1 = \{0\}$ or $\Omega_2 = \{0\}$.

Theorem (Yamasaki 2014)

Let Ω be an irreducible homogeneous cone.
Then Ω is symmetric if and only if

$$\{\deg \Delta_1, \ldots, \deg \Delta_r\} = \{\deg \Delta_1^*, \ldots, \deg \Delta_r^*\} = \{1, \ldots, r\}.$$

Example

$\boldsymbol{V = \mathcal{S}_3}$
$\boldsymbol{\Omega = \mathcal{S}_3^+}$ and $\boldsymbol{\Omega^* = \mathcal{S}_3^+}$ (symmetric cone)

$$\boldsymbol{V = \left\{ x = \begin{pmatrix} x_1 & x_{21} & x_{31} \\ x_{21} & x_2 & x_{32} \\ x_{31} & x_{32} & x_3 \end{pmatrix} ;\ x_i, x_{kj} \in \mathbb{R} \right\}}$$

The basic relative invariants are described as

$$\begin{aligned} \boldsymbol{\Delta_1(x)} &= \boldsymbol{x_1}, & \boldsymbol{\Delta_1^*(x)} &= \boldsymbol{\det x}, \\ \boldsymbol{\Delta_2(x)} &= \boldsymbol{x_1 x_2 - x_{21}^2}, & \boldsymbol{\Delta_2^*(x)} &= \boldsymbol{x_3 x_2 - x_{32}^2}, \\ \boldsymbol{\Delta_3(x)} &= \boldsymbol{\det x}, & \boldsymbol{\Delta_3^*(x)} &= \boldsymbol{x_3}. \end{aligned}$$

The multiplier matrices is given as

$$\boldsymbol{\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \sigma_* = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.}$$

Main theorem

Theorem 2

Suppose that Ω is irreducible. Then Ω is symmetric if and only if

$$\begin{aligned}
&(1)\ \mathrm{Re}\,\frac{\Delta_j(w)}{\Delta_{j-1}(w)} > 0 \quad \text{for any } w \in \Omega + iV, \\
&(2)\ \mathrm{Re}\,\frac{\Delta_j^*(w^*)}{\Delta_{j+1}^*(w^*)} > 0 \quad \text{for any } w^* \in \Omega^* + iV
\end{aligned} \qquad (j = 1, \ldots, r),$$

where we put $\Delta_0(w) = 1$ and $\Delta_{r+1}^*(w) = 1$.

Key proposition

Proposition 3

Let τ be a lower triangular matrix of integer elements with ones on the main diagonal. Assume

$$\operatorname{Re} \Delta^{\underline{e}_j \tau}(w) > 0 \quad (j = 1, \ldots, r)$$

for any $w \in T_\Omega$. Then one has $\tau = \sigma^{-1}$.

From the condition (1), we obtain

$$\sigma^{-1} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix}$$

Algorithm for calculating multiplier matrix

$d_{kj} := \dim \mathcal{V}_{kj}$ $(1 \le j < k \le r)$
$d_i := {}^t(0, \ldots, 0, d_{i+1,i}, \ldots, d_{ri})$ $(i = 1, \ldots, r-1)$.
For $i = 1, \ldots, r-1$, we define $l_i^{(j)} = {}^t(l_{1i}^{(j)}, \ldots, l_{ri}^{(j)})$ $(j = i, \ldots, r)$

$$
\begin{aligned}
l_i^{(i)} &:= d_i && (k = i), \\
l_i^{(k+1)} &:= \begin{cases} l_i^{(j)} - d_k & (l_{ii}^{(j)} > 0), \\ l_i^{(j)} & (l_{ii}^{(j)} = 0) \end{cases} && (k > i).
\end{aligned}
$$

Moreover we set $\varepsilon^{[i]} = {}^t(\varepsilon_{i+1,i}, \ldots, \varepsilon_{ri}) \in \{0,1\}^{r-i}$
$(i = 1, \ldots, r-1)$ by

$$
\varepsilon_{ki} = \begin{cases} 1 & \text{if } l_{ik}^{(k)} > 0, \\ 0 & \text{if } l_{ik}^{(k)} = 0 \end{cases} \quad (k = i+1, \ldots, r).
$$

Algorithm for calculating multiplier matrix

Then σ is given as

$$\sigma = \mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1,$$

where

$$\mathcal{E}_i := \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^{[i]} & I_{n-i} \end{pmatrix}.$$

σ^{-1} is described as

$$\begin{aligned} \sigma^{-1} &= (\mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1)^{-1} \\ &= \mathcal{E}_1^{-1}\mathcal{E}_2^{-1}\cdots\mathcal{E}_{r-1}^{-1} \\ &= \begin{pmatrix} 1 & & & \\ -\varepsilon_{21} & 1 & & \\ \vdots & & \ddots & \\ -\varepsilon_{r1} & -\varepsilon_{r2} & \cdots & 1 \end{pmatrix}. \end{aligned}$$

Sketch of the proof

Since

$$\begin{pmatrix} 1 & & & \\ -\varepsilon_{21} & 1 & & \\ \vdots & & \ddots & \\ -\varepsilon_{r1} & -\varepsilon_{r2} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix},$$

we have $(\varepsilon_{21}, \varepsilon_{31}, \ldots, \varepsilon_{r1}) = (1, 0, \ldots, 0)$.
Let $l_1^{(1)} = {}^t(d_{21}, \ldots, d_{r1})$.
By $\varepsilon_{21} = 1$, one has $d_{21} > 0$ and

$$l_1^{(2)} = \begin{pmatrix} d_{21} \\ d_{31} - d_{32} \\ \vdots \\ d_{r1} - d_{r2} \end{pmatrix}$$

By $\varepsilon_{31} = 0$, one has $d_{31} - d_{32} = 0$.
Similarly $\varepsilon_{k1} = 0$ implies $d_{k1} - d_{k2} = 0 \quad (k > 3)$.

Sketch of the proof

Repetition of this arguments implies that

$$d_{k1} = d_{k2} = \cdots = d_{k,k-1} \quad (k = 2, \ldots, r-1).$$

Similarly by σ_*, we obtain

$$d_{j+1,j} = d_{j+2,j} = \cdots = d_{rj} \quad (j = 1, \ldots, r-1).$$

Thus there exists the common number $d = d_{kj} > 0 \quad (j < k)$.
By the following theorem, Ω needs to be symmetric.

Theorem (Vinberg 1965)

If $\dim \mathcal{V}_{kj} =$ (const), then Ω is a symmetric cone.

Counter example (Ishi–Nomura 2008)

$$V := \left\{ x = \begin{pmatrix} x_1 I_n & a I_n & \mathrm{b} \\ a I_n & x_2 I_n & \mathrm{c} \\ {}^t\mathrm{b} & {}^t\mathrm{c} & x_3 \end{pmatrix};\ \begin{array}{l} x_i, a \in \mathbb{R} \\ \mathrm{b}, \mathrm{c} \in \mathbb{R}^n \end{array} \right\},$$

$$\Omega := \{ x \in V;\ x \text{ is positive definite} \}.$$

$\Delta_1(x), \ldots, \Delta_3(x)$ are given as

$$\begin{aligned} &\Delta_1(x) = x_1, \qquad \Delta_2(x) = x_1 x_2 - a^2, \\ &\Delta_3(x) = x_1 x_2 x_3 + 2a \langle \mathrm{b} \,|\, \mathrm{c} \rangle - x_3 a^2 - x_2 \|\mathrm{b}\|^2 - x_1 \|\mathrm{c}\|^2. \end{aligned}$$

Ω is not a symmetric cone if $n \geq 2$, but we have

$$\mathrm{Re}\, \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (w \in \Omega + iV,\ k = 1, 2, 3).$$

In this case $\dim V_{21} = 1$, $\dim V_{31} = n$, $\dim V_{32} = n$.
Thus we obtain

$$\sigma^{-1} = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{pmatrix}.$$

References

[1] H. Ishi, *On symplectic representations of normal j-algebras and their application to Xu's realizations of Siegel domains*, Differ. Geom. Appl., **24** (2006), 588–612.

[2] H. Ishi and T. Nomura, *Tube domain and an orbit of a complex triangular group*, Math. Z., **259** (2008), 697–711.

[3] H. Nakashima, *Basic relative invariants of homogeneous cones*, Journal of Lie Theory **24** (2014), 1013–1032.

[4] H. Nakashima, *Characterization of symmetric cones by means of the basic relative invariants*, submitting.

[5] E. B. Vinberg, *The structure of the group of automorphisms of a homogeneous convex cone*, Trans. Moscow Math. Soc., **13** (1965), 63–93.

[6] T. Yamasaki, *Studies on homogeneous cones and the basic relative invariants through skeleton*, Doctoral thesis admitted to Kyushu university (2014)